Towards a Direct Method for the Analyticity of the Pressure for Certain Classical Unbounded Spin Systems

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Abstract

The aim of this paper is to study direct methods for the analyticity of the pressure for certain classical unbounded spin models. We provide a representation in terms of the Witten Laplacian on one-forms of the nth-derivative of the pressure as function of some order parameter t. The technique involves the formula for the covariance introduced by B. Helffer and J. Sjostrand.

1 Introduction

As already mentioned in [66], The methods for investigating critical phenomena for certain physical systems took an interesting direction when powerful and sophisticated PDE techniques were introduced. The methods are generally based on the analysis of suitable differential operators

$$\mathbf{W}_{\Phi}^{(0)} = \left(-\mathbf{\Delta}+\frac{|\mathbf{\nabla}\Phi|^2}{4}-\frac{\mathbf{\Delta}\Phi}{2}\right)$$

and

$$\mathbf{W}_{\Phi}^{(1)} = -\mathbf{\Delta}+\frac{|\mathbf{\nabla}\Phi|^2}{4}-\frac{\mathbf{\Delta}\Phi}{2}+\mathbf{Hess}\Phi.$$

These are in some sense deformations of the standard Laplace Beltrami operator. They are commonly called Witten Laplacians, and were first introduced by Edward Witten, [18] in 1982 in the context of Morse theory for the study of topological invariants of compact Riemannian manifolds. In 1994, Bernard Helffer and Johannes Sjöstrand [8] introduced two elliptic differential operators

$$A_{\Phi}^{(0)} := -\mathbf{\Delta}+\mathbf{\nabla}\Phi\cdot\mathbf{\nabla}$$

and

$$A_{\Phi}^{(1)} := -\mathbf{\Delta}+\mathbf{\nabla}\Phi\cdot\mathbf{\nabla}+\mathbf{Hess}\Phi$$

sometimes called Helffer-Sjöstrand operators serving to get direct method for the study of integrals and operators in high dimensions of the type that appear in Statistical Mechanics and Euclidean Field Theory. In 1996, Johannes Sjöstrand [13] observed that these so-called Helffer-Sjöstrand operators were in fact equivalent to Witten's Laplacians. Since then, there have been significant advances in the use of these Laplacians to study the thermodynamic behavior of quantities related to the Gibbs measure $Z^{-1}e^{-\Phi}dx$.

Numerous techniques have been developed in the study of integrals associated with the equilibrium Gibbs state for certain unbounded spins systems. One of the most striking results is an exact formula for the covariance of two functions in terms of the Witten Laplacian on one forms leading to sophisticated methods for estimating the correlation functions of a random field. As mentioned in [10], this formula is in some sense a stronger and more flexible version of the Brascamp-Lieb inequality [1]. The formula may be written as follow:

$$\mathbf{cov}(f,g) = \int \left(A_{\Phi}^{(1)^{-1}} \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g \right) e^{-\Phi(x)} dx. \tag{1}$$

We attempt in these notes, to study a direct method for the analyticity of the pressure for certain classical convex unbounded spin systems. It is central in Statistical Mechanics to study the differentiability or even the analyticity of the pressure with respect to some distinguished thermodynamic parameters such as temperature, chemical potential or external field. In fact the analytic behavior of the pressure is the classical thermodynamic indicator for the absence or existence of phase transition. The most famous result on the analyticity of the pressure is the circle theorem of Lee and Yang [28]. This theorem asserts the following: consider a $\{-1,1\}-$valued spin system with ferromagnetic pair interaction and external field h and regard the quantity $z = e^h$ as a complex parameter, then all zeroes of all partition functions (with free boundary condition), considered as functions of z lie in the complex unit circle. This theorem readily implies that the pressure is an analytic function of h in the region $h > 0$ and $h < 0$. Heilmann [29] showed that the assumption of pair interaction is necessary. A transparent approach to the circle theorem was found by Asano [30] and developed further by Ruelle [31],[32], Slawny [33], and Gruber et al [34]. Griffiths [35] and Griffiths-Simon [36] found a method of extending the Lee-Yang theorem to real-valued spin systems with a particular type of a priory measure. Newman [37] proved the Lee-Yang theorem for every a priory measure which satisfies this theorem in the particular case of no interaction. Dunlop [38],[39] studied the zeroes of the partition functions for the plane rotor model. A general Lee-Yang theorem for multicomponent systems was finally proved by Lieb and Sokal [40]. For further references see Glimm and Jaffe [41].
The Lee-Yang theorem and its variants depend on the ferromagnetic character of the interaction. There are various other way of proving the infinite differentiability or the analyticity of the pressure for (ferromagnetic and non ferromagnetic) systems at high temperatures, or at low temperatures, or at large external fields. Most of these take advantage of a sufficiently rapid decay of correlations and /or cluster expansion methods. Here is a small sample of relevant

references. Bricmont, Lebowitz and Pfister [42], Dobroshin [43], Dobroshin and Sholsman [44],[45], Duneau et al [46],[47],[48], Glimm and Jaffe [41],[49], Israel [50], Kotecky and Preiss [51], Kunz [52], Lebowitz [53],[54], Malyshev [55], Malychev and Milnos [56] and Prakash [57]. M. Kac and J.M. Luttinger [58] obtained a formula for the pressure in terms of irreducible distribution functions.

In this present study, we propose a new way of analyzing the analyticity of the pressure for certain unbounded models through a representation by means of the Witten Laplacians of the remainder of the Taylor series expansion. The methods known up to now rely on complicated indirect arguments.

2 Towards the analyticity of the Pressure

Let Λ be a finite domain in $\mathbb{Z}^d$ $(d \geq 1)$ and consider the Hamiltonian of the phase space given by,

$$\Phi(x) = \Phi_\Lambda(x) = \frac{x^2}{2} + \Psi(x), \qquad x \in \mathbb{R}^\Lambda. \tag{2}$$

where

$$|\partial^\alpha \boldsymbol{\nabla} \Psi| \leq C_\alpha, \qquad \forall \alpha \in \mathbb{N}^{|\Lambda|}, \tag{3}$$

$$\mathbf{Hess}\Phi(x) \geq \delta_o, \qquad 0 < \delta_o < 1. \tag{4}$$

Let g is a smooth function on $\mathbb{R}^\Gamma$ with lattice support $S_g = \Gamma$. We identified with $\tilde{g}$ defined on $\mathbb{R}^\Lambda$ by

$$\tilde{g}(x) = g(x_\Gamma) \quad \text{where } x = (x_i)_{i\in\Lambda} \quad \text{and } x_\Gamma = (x_i)_{i\in\Gamma} \tag{5}$$

and satisfying

$$|\partial^\alpha \boldsymbol{\nabla} g| \leq C_\alpha \qquad \forall \alpha \in \mathbb{N}^{|\Gamma|} \tag{6}$$

Under the additional assumptions that Ψ is compactly supported in $\mathbb{R}^\Lambda$ and g is compactly supported in $\mathbb{R}^\Gamma$, it was proved in [66] (see also [8]) that the equation

$$\begin{cases} -\boldsymbol{\Delta} f + \boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla} f = g - \langle g \rangle \\ \langle f \rangle_{L^2(\mu)} = 0 \end{cases}$$

has a unique smooth solution satisfying $\boldsymbol{\nabla}^k f(x) \to 0$ as $|x| \to \infty$ for every $k \geq 1$.

Recall also that $\boldsymbol{\nabla} f$ is a solution of the system

$$(-\boldsymbol{\Delta} + \boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla})\, \boldsymbol{\nabla} f + \mathbf{Hess}\Phi \boldsymbol{\nabla} f = \boldsymbol{\nabla} g \quad \text{in } \mathbb{R}^\Lambda. \tag{7}$$

As in [66] and [8], these assumptions will be relaxed later on.

Let

$$\Phi^t_\Lambda(x) = \Phi(x) - tg(x), \tag{8}$$

where $x = (x_i)_{i\in\Lambda}$, and assume additionally that g satisfies

$$\mathbf{Hess} g \leq C. \tag{9}$$

We consider the following perturbation

$$\theta_\Lambda(t) = \log\left[\int dx e^{-\Phi^t_\Lambda(x)}\right]. \tag{10}$$

Denote by

$$Z_t = \int dx e^{-\Phi^t_\Lambda(x)} \tag{11}$$

and

$$< \cdot >_{t,\Lambda} = \frac{\int \cdot \, dx e^{-\Phi^t_\Lambda(x)}}{Z_t}. \tag{12}$$

3 Parameter Dependency of the Solution

From the assumptions made on Φ and g, it is easy to see that there exists $T > 0$ such that or every $t \in [0, T)$, $\Phi^t_\Lambda(x)$ satisfies all the assumptions required for the solvability, regularity and asymptotic behavior of the solution $f(t)$ associated with the potential $\Phi^t_\Lambda(x)$. Thus, each $t \in [0, T)$ is associated with a unique $C^\infty-$solution, $f(t)$ of the equation

$$\begin{cases} A^{(0)}_{\Phi^t_\Lambda} f(t) = g - \langle g \rangle_{L^2(\mu)} \\ \langle f(t) \rangle_{L^2(\mu)} = 0. \end{cases}$$

Hence,

$$A^{(1)}_{\Phi^t_\Lambda} \mathbf{v}(t) = \boldsymbol{\nabla} g \tag{13}$$

where $\mathbf{v}(t) = \boldsymbol{\nabla} f(t)$. Notice that the map

$$t \longmapsto \mathbf{v}(t)$$

is well defined and

$$\{\mathbf{v}(t) : t \in [0, T)\}$$

is a family of smooth solutions on $\mathbb{R}^\Lambda$ satisfying

$$\partial^\alpha \mathbf{v}(t) \to 0 \quad \text{as } |x| \to \infty \qquad \forall \alpha \in \mathbb{N}^{|\Lambda|} \text{ and for each } t \in [0, T)$$

and corresponding to the family of potential

$$\left\{\Phi^t_\Lambda : t \in [0, T)\right\}. \tag{14}$$

Let us now verify that $\mathbf{v}$ is a smooth function of $t \in (0, T)$.We need to prove that for each $t \in (0, T)$, the limit

$$\lim_{\varepsilon \to 0} \frac{\mathbf{v}(t + \varepsilon) - \mathbf{v}(t)}{\varepsilon}$$

exists. Let

$$\mathbf{v}^\varepsilon(t) = \frac{\mathbf{v}(t + \varepsilon) - \mathbf{v}(t)}{\varepsilon}.$$

We use a technique based on regularity estimates to get a uniform control of $\mathbf{v}^\varepsilon(t)$ with respect to ε.

With ε small enough, we have

$$\begin{aligned} 0 &= -\mathbf{\Delta}\left[\frac{\mathbf{v}(t+\varepsilon)-\mathbf{v}(t)}{\varepsilon}\right]+\frac{\mathbf{\nabla}\Phi^{t+\varepsilon}\cdot\mathbf{\nabla}\mathbf{v}(t+\varepsilon)-\mathbf{\nabla}\Phi^{t}\cdot\mathbf{\nabla}\mathbf{v}(t)}{\varepsilon} \\ &\quad+\frac{\mathbf{Hess}\Phi^{t+\varepsilon}\mathbf{v}(t+\varepsilon)-\mathbf{Hess}\Phi^{t}\mathbf{v}(t)}{\varepsilon}. \end{aligned}$$

Equivalently,

$$\begin{aligned} &-\mathbf{\Delta}\left[\frac{\mathbf{v}(t+\varepsilon)-\mathbf{v}(t)}{\varepsilon}\right]+\frac{\mathbf{\nabla}\Phi^{t+\varepsilon}\cdot\mathbf{\nabla}\left[\mathbf{v}(t+\varepsilon)-\mathbf{v}(t)\right]}{\varepsilon} \\ &+\mathbf{Hess}\Phi^{t+\varepsilon}\left(\frac{\mathbf{v}(t+\varepsilon)-\mathbf{v}(t)}{\varepsilon}\right) \\ =\;& -\left(\frac{\mathbf{Hess}\Phi^{t+\varepsilon}-\mathbf{Hess}\Phi^{t}}{\varepsilon}\right)\mathbf{v}(t)-\left(\frac{\mathbf{\nabla}\Phi^{t+\varepsilon}-\mathbf{\nabla}\Phi^{t}}{\varepsilon}\right)\cdot\mathbf{\nabla}\mathbf{v}(t) \end{aligned}$$

and

$$\begin{aligned} &-\mathbf{\Delta}\mathbf{v}^\varepsilon(t)+\mathbf{\nabla}\Phi^{t+\varepsilon}\cdot\mathbf{\nabla}\mathbf{v}^\varepsilon(t)+\mathbf{Hess}\Phi^{t+\varepsilon}\mathbf{v}^\varepsilon(t) \\ =\;& -\left(\frac{\mathbf{Hess}\Phi^{t+\varepsilon}-\mathbf{Hess}\Phi^{t}}{\varepsilon}\right)\mathbf{v}(t)-\left(\frac{\mathbf{\nabla}\Phi^{t+\varepsilon}-\mathbf{\nabla}\Phi^{t}}{\varepsilon}\right)\cdot\mathbf{\nabla}\mathbf{v}(t). \end{aligned}$$

Let $\mathbf{w}(t)$ be the unique $C^\infty-$solution of the system

$$-\mathbf{\Delta}\mathbf{w}(t)+\mathbf{\nabla}\Phi^{t}\cdot\mathbf{\nabla}\mathbf{w}(t)+\mathbf{Hess}\Phi^{t}\mathbf{w}(t)=\mathbf{Hess}g\mathbf{v}(t)-\mathbf{\nabla}g\cdot\mathbf{\nabla}\mathbf{v}(t). \tag{15}$$

Combining the last two systems above, we get

$$\begin{aligned} &-\mathbf{\Delta}\left[\mathbf{w}(t)-\mathbf{v}^\varepsilon(t)\right]+\mathbf{\nabla}\Phi^{t}\cdot\mathbf{\nabla}\left[\mathbf{w}(t)-\mathbf{v}^\varepsilon(t)\right]+\mathbf{Hess}\Phi^{t}\left[\mathbf{w}(t)-\mathbf{v}^\varepsilon(t)\right] \\ &=\mathbf{Hess}g\mathbf{v}(t)-\mathbf{\nabla}g\cdot\mathbf{\nabla}\mathbf{v}(t)+\left(\frac{\mathbf{Hess}\Phi^{t+\varepsilon}-\mathbf{Hess}\Phi^{t}}{\varepsilon}\right)\mathbf{v}(t) \\ &\quad+\left(\frac{\mathbf{\nabla}\Phi^{t+\varepsilon}-\mathbf{\nabla}\Phi^{t}}{\varepsilon}\right)\cdot\mathbf{\nabla}\mathbf{v}(t)+\left(\mathbf{\nabla}\Phi^{t+\varepsilon}-\mathbf{\nabla}\Phi^{t}\right)\cdot\mathbf{\nabla}\mathbf{v}^\varepsilon(t) \\ &\quad+\left(\mathbf{Hess}\Phi^{t+\varepsilon}-\mathbf{Hess}\Phi^{t}\right)\mathbf{v}^\varepsilon(t). \end{aligned} \tag{16}$$

Now using the unitary transformation U_{Φ^t}, we get

$$\begin{aligned} &\left(-\mathbf{\Delta}+\frac{\left|\mathbf{\nabla}\Phi^{t}\right|^2}{4}-\frac{\mathbf{\Delta}\Phi^{t}}{2}\right)\left(\mathbf{w}(t)-\mathbf{v}^\varepsilon(t)\right)e^{-\Phi^t/2} \\ &\quad+\mathbf{Hess}\Phi^{t}\left(\mathbf{w}(t)-\mathbf{v}^\varepsilon(t)\right)e^{-\Phi^t/2} \\ &=o_\varepsilon(1)e^{-\Phi^t/2}+\left[\left(\mathbf{\nabla}\Phi^{t+\varepsilon}-\mathbf{\nabla}\Phi^{t}\right)\cdot\mathbf{\nabla}\mathbf{v}^\varepsilon(t)\right. \\ &\quad\left.+\left(\mathbf{Hess}\Phi^{t+\varepsilon}-\mathbf{Hess}\Phi^{t}\right)\mathbf{v}^\varepsilon(t)\right]e^{-\Phi^t/2} \end{aligned} \tag{17}$$

Next, we propose to estimate the last two terms of the right hand side of this equation.

Again using the unitary transformation $U_{\Phi^{t+\varepsilon}}$, we reduce the system

$$\begin{aligned} -\boldsymbol{\Delta}\mathbf{v}^{\varepsilon}(t) + \boldsymbol{\nabla}\Phi^{t+\varepsilon}\cdot\boldsymbol{\nabla}\mathbf{v}^{\varepsilon}(t) + \mathbf{Hess}\Phi^{t+\varepsilon}\mathbf{v}^{\varepsilon}(t) \\ = -\left(\frac{\mathbf{Hess}\Phi^{t+\varepsilon} - \mathbf{Hess}\Phi^{t}}{\varepsilon}\right)\mathbf{v}(t) \\ -\left(\frac{\boldsymbol{\nabla}\Phi^{t+\varepsilon} - \boldsymbol{\nabla}\Phi^{t}}{\varepsilon}\right)\cdot\boldsymbol{\nabla}\mathbf{v}(t) \end{aligned} \tag{18}$$

into

$$\begin{aligned} \left(-\boldsymbol{\Delta}+\frac{|\boldsymbol{\nabla}\Phi^{t+\varepsilon}|^2}{4} - \frac{\boldsymbol{\Delta}\Phi^{t+\varepsilon}}{2}\right)\mathbf{V}^{\varepsilon} + \mathbf{Hess}\Phi^{t+\varepsilon}\mathbf{V}^{\varepsilon} = \\ -\left(\frac{\mathbf{Hess}\Phi^{t+\varepsilon} - \mathbf{Hess}\Phi^{t}}{\varepsilon}\right)\mathbf{v}(t)e^{-\Phi^{t+\varepsilon}/2} \\ -\left(\frac{\boldsymbol{\nabla}\Phi^{t+\varepsilon} - \boldsymbol{\nabla}\Phi^{t}}{\varepsilon}\right)\cdot\boldsymbol{\nabla}\mathbf{v}(t)e^{-\Phi^{t+\varepsilon}/2} \end{aligned} \tag{19}$$

where $\mathbf{V}^{\varepsilon} = \mathbf{v}^{\varepsilon}(t)e^{-\Phi^{t+\varepsilon}/2}$. Taking scalar product with $\mathbf{V}^{\varepsilon}$ on both sides of this last equality and integrating, we get

$$\begin{aligned} \left\|\left(\partial_x+\frac{\boldsymbol{\nabla}\Phi^{t+\varepsilon}}{2}\right)\mathbf{V}^{\varepsilon}\right\|_{L^2}^2 + \int \mathbf{Hess}\Phi^{t+\varepsilon}\mathbf{V}^{\varepsilon}\cdot\mathbf{V}^{\varepsilon}dx = \\ -\int\left[\left(\frac{\mathbf{Hess}\Phi^{t+\varepsilon} - \mathbf{Hess}\Phi^{t}}{\varepsilon}\right)\mathbf{v}(t)e^{-\Phi^{t+\varepsilon}/2}\right]\cdot\mathbf{V}^{\varepsilon}dx \\ -\left[\int\left(\frac{\boldsymbol{\nabla}\Phi^{t+\varepsilon} - \boldsymbol{\nabla}\Phi^{t}}{\varepsilon}\right)\cdot\boldsymbol{\nabla}\mathbf{v}(t)e^{-\Phi^{t+\varepsilon}/2}\right]\cdot\mathbf{V}^{\varepsilon}dx \end{aligned} \tag{20}$$

Now using the uniform strict convexity on the left hand side and Cauchy-Schwartz on the right hand side, we obtain

$$\|\mathbf{V}^{\varepsilon}\|_{B^0} \le C \qquad \text{for small enough } \varepsilon. \tag{21}$$

We then deduce that

$$\left(-\boldsymbol{\Delta}+\frac{|\boldsymbol{\nabla}\Phi^{t+\varepsilon}|^2}{4}\right)\mathbf{V}^{\varepsilon} = \tilde{q}_{\varepsilon} \tag{22}$$

where

$$\begin{aligned} \tilde{q}_{\varepsilon} = -\left(\frac{\mathbf{Hess}\Phi^{t+\varepsilon} - \mathbf{Hess}\Phi^{t}}{\varepsilon}\right)\mathbf{v}(t)e^{-\Phi^{t+\varepsilon}/2} - \left(\frac{\boldsymbol{\nabla}\Phi^{t+\varepsilon} - \boldsymbol{\nabla}\Phi^{t}}{\varepsilon}\right)\cdot\boldsymbol{\nabla}\mathbf{v}(t)e^{-\Phi^{t+\varepsilon}/2} \\ +\frac{\boldsymbol{\Delta}\Phi^{t+\varepsilon}}{2}\mathbf{V}^{\varepsilon} - \mathbf{Hess}\Phi^{t+\varepsilon}\mathbf{V}^{\varepsilon} \end{aligned} \tag{23}$$

is bounded in B^0 uniformly with respect to ε for ε small enough.
Now taking scalar product with $\mathbf{V}^{\varepsilon}$ on both sides of (28) and integrating by parts, we obtain

$$\|\boldsymbol{\nabla}\mathbf{V}^{\varepsilon}\|_{L^2}^2 + \left\|\frac{|\boldsymbol{\nabla}\Phi^{t+\varepsilon}|}{2}\mathbf{V}^{\varepsilon}\right\|_{L^2}^2 \le \|\tilde{q}_{\varepsilon}\|_{L^2}\|\mathbf{V}^{\varepsilon}\|_{L^2} \tag{24}$$

It follows that $\mathbf{V}^\varepsilon$ is uniformly bounded with respect to ε in $B^1_{\Phi^{t+\varepsilon}}$ for ε small enough.

Next, observe that

$$\left(-\boldsymbol{\Delta}+\frac{|\boldsymbol{\nabla}\Phi^t|^2}{4}\right)\mathbf{V}^\varepsilon = \hat{q}_\varepsilon \tag{25}$$

where

$$\hat{q}_\varepsilon = \tilde{q}_\varepsilon - \frac{|\boldsymbol{\nabla}\Phi^{t+\varepsilon} - \boldsymbol{\nabla}\Phi^t|^2}{4}\mathbf{V}^\varepsilon + \frac{(\boldsymbol{\nabla}\Phi^{t+\varepsilon} - \boldsymbol{\nabla}\Phi^t)\cdot\boldsymbol{\nabla}\Phi^t}{2}\mathbf{V}^\varepsilon \tag{26}$$

is uniformly bounded in B^0 with respect to ε for small enough ε. Using regularity, it follows that for small enough ε, $\mathbf{V}^\varepsilon$ is uniformly bounded in $B^2_{\Phi^t}$ with respect to ε.This implies that $\hat{q}_\varepsilon$ is uniformly bounded in $B^1_{\Phi^t}$ for ε small enough. Again, we can continue by a bootstrap argument to consequently get that for ε small enough, $\mathbf{V}^\varepsilon$ is uniformly bounded in $B^k_{\Phi^t}$ for any k.
It is then clear that for small enough ε, the right hand sides of (23) is $\mathcal{O}(\varepsilon)$ in B^0 and consequently, using the same argument as above, we get that $(\mathbf{w}(t) - \mathbf{v}^\varepsilon(t))\, e^{-\Phi^t/2}$ is $\mathcal{O}(\varepsilon)$ in $B^2_{\Phi^t}$; again iterating the regularity argument, we obtain that for small enough ε, $(\mathbf{w}(t) - \mathbf{v}^\varepsilon(t))\, e^{-\Phi^t/2}$ is $\mathcal{O}(\varepsilon)$ $B^k_{\Phi^t}$ for every k. We have proved:

Proposition 1 *Under the above on Φ and g, there exists $T > 0$ so that for each $t \in (0,T)$, $\mathbf{v}^\varepsilon(t)$ converges to $\mathbf{w}(t)$ in C^∞.*

Remark 2 *The proposition establishes that $\mathbf{v}(t)$ is differentiable in t and $\dfrac{d}{dt}\mathbf{v}(t)$ is given by the unique $C^\infty-$solution $\mathbf{w}(t)$ of the system*

$$-\boldsymbol{\Delta}\mathbf{w}(t) + \boldsymbol{\nabla}\Phi^t\cdot\boldsymbol{\nabla}\mathbf{w}(t) + \mathbf{Hess}\Phi^t\mathbf{w}(t) = \mathbf{Hess}g\mathbf{v}(t) - \boldsymbol{\nabla}g\cdot\boldsymbol{\nabla}\mathbf{v}(t). \tag{27}$$

Iterating this argument, we easily get that, $\mathbf{v}(t)$ is smooth in $t \in (0,T)$.

Now we are ready for the following:

4 Formula for $\theta^{(n)}(t)$

For an arbitrary suitable function $f(t) = f(t,w)$

$$\frac{\partial}{\partial t} < f(t) >_{t,\Lambda} = < f\,'(t) >_{t,\Lambda} + \mathbf{cov}(f,g). \tag{28}$$

Hence,

$$\frac{\partial}{\partial t} < f(t) >_{t,\Lambda} = < f\,'(t) >_{t,\Lambda} + < A^{(1)^{-1}}_{\Phi^t}(\boldsymbol{\nabla}f)\cdot\boldsymbol{\nabla}g >_{t,\Lambda}. \tag{29}$$

Let

$$A_g f := A^{(1)^{-1}}_{\Phi^t}(\boldsymbol{\nabla}f)\cdot\boldsymbol{\nabla}g. \tag{30}$$

Thus,

$$\frac{\partial}{\partial t} < f(t) >_{t,\Lambda} = < \left(\frac{\partial}{\partial t} + A_g\right) f >_{t,\Lambda} . \tag{31}$$

The linear operator $\frac{\partial}{\partial t} + A_g$ will be denoted by H_g.

$$\begin{aligned} \theta'_\Lambda(t) &= < g >_{t,\Lambda} \\ &= < \left(\frac{\partial}{\partial t} + A_g\right)^0 g >_{t,\Lambda} \\ &= < H_g^0 g >_{t,\Lambda}; \end{aligned}$$

$$\begin{aligned} \theta''_\Lambda(t) &= \frac{\partial}{\partial t} < g >_{t,\Lambda} \\ &= < A_{\Phi^t}^{(1)^{-1}} (\boldsymbol{\nabla} g) \cdot \boldsymbol{\nabla} g >_{t,\Lambda} \\ &= < \left(\frac{\partial}{\partial t} + A_g\right) g >_{t,\Lambda}; \end{aligned}$$

$$\begin{aligned} \theta'''_\Lambda(t) &= \frac{\partial}{\partial t} < A_{\Phi^t}^{(1)^{-1}} (\boldsymbol{\nabla} g) \cdot \boldsymbol{\nabla} g >_{t,\Lambda} \\ &= < \frac{\partial}{\partial t} \left(A_{\Phi^t}^{(1)^{-1}} (\boldsymbol{\nabla} g) \cdot \boldsymbol{\nabla} g\right) >_{t,\Lambda} \\ + &< \left(A_{\Phi^t}^{(1)^{-1}} \boldsymbol{\nabla} \left(A_{\Phi^t}^{(1)^{-1}} (\boldsymbol{\nabla} g) \cdot \boldsymbol{\nabla} g\right)\right) \cdot \boldsymbol{\nabla} g >_{t,\Lambda} \\ &= < \left(\frac{\partial}{\partial t} + A_g\right)^2 g >_{t,\Lambda} . \end{aligned}$$

By induction it is easy to see that

$$\begin{aligned} \theta_\Lambda^{(n)}(t) &= < \left(\frac{\partial}{\partial t} + A_g\right)^{n-1} g >_{t,\Lambda} \qquad (\forall n \geq 1) \\ &= < H_g^{(n-1)} g >_{t,\Lambda} . \end{aligned}$$

Next, we propose to find a simpler formula for $\theta_\Lambda^{(n)}(t)$ that only involves A_g.

$$\begin{aligned} H_g g &= A_{\Phi^t}^{(1)^{-1}} (\boldsymbol{\nabla} g) \cdot \boldsymbol{\nabla} g \\ &= A_g g \end{aligned}$$

$$H_g^2 g = \frac{\partial}{\partial t} \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g + \left(A_{\Phi^t}^{(1)^{-1}} \boldsymbol{\nabla} \left(A_{\Phi^t}^{(1)^{-1}} (\boldsymbol{\nabla} g) \cdot \boldsymbol{\nabla} g\right)\right) \cdot \boldsymbol{\nabla} g \tag{32}$$

where f satisfies the equation

$$\boldsymbol{\nabla} f = A_{\Phi^t}^{(1)^{-1}} (\boldsymbol{\nabla} g) . \tag{33}$$

With $\mathbf{v}(t) = \boldsymbol{\nabla} f$, as before, we get

$$\frac{\partial}{\partial t}\boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g = A_{\Phi^t}^{(1)^{-1}} \left(\mathbf{Hess} g \mathbf{v}(t) - \boldsymbol{\nabla} g \cdot \boldsymbol{\nabla}\mathbf{v}(t)\right) \cdot \boldsymbol{\nabla} g$$

and H_g^2 becomes

$$\begin{aligned} H_g^2 g &= A_{\Phi^t}^{(1)^{-1}} \left[\left(\mathbf{Hess} g \mathbf{v}(t) - \boldsymbol{\nabla} g \cdot \boldsymbol{\nabla}\mathbf{v}(t)\right) + \boldsymbol{\nabla} \left(A_{\Phi^t}^{(1)^{-1}} (\boldsymbol{\nabla} g) \cdot \boldsymbol{\nabla} g \right) \right] \cdot \boldsymbol{\nabla} g \\ &= A_{\Phi^t}^{(1)^{-1}} 2 \boldsymbol{\nabla} \left(A_g g\right) \cdot \boldsymbol{\nabla} g \\ &= 2 A_g^2 g. \end{aligned}$$

Proposition 3 *If*

$$\theta_\Lambda(t) = \log \left[\int dx e^{-\Phi^t(x)} \right]$$

where

$$\Phi^t(x) = \Phi_\Lambda(x) - t g(x)$$

is as above then $\theta_\Lambda^{(n)}(t)$*, the* $nth-$ *derivative of* $\theta_\Lambda(t)$ *is given by the formula*

$$\theta'_\Lambda(t) =< g >_{t,\Lambda},$$

and for $n \geq 1$

$$\theta_\Lambda^{(n)}(t) = (n-1)! < A_g^{n-1} g >_{t,\Lambda} .$$

Proof. We have already established that

$$\theta_\Lambda^{(n)}(t) =< H_g^{n-1} g >_{t,\Lambda} \qquad for\ n \geq 1.$$

It then only remains to prove that

$$H_g^{n-1} g = (n-1)! A_g^{n-1} g \qquad for\ n \geq 1.$$

The result is already established above for $n = 1, 2, 3,$. By induction, assume that

$$H_g^{n-1} g = (n-1)! A_g^{n-1} g \ .$$

if n is replaced by $\tilde{n} \leq n$.

$$\begin{aligned} H_g^n g &= \left(\frac{\partial}{\partial t} + A_g \right) \left((n-1)! A_g^{n-1} g \right) \\ &= (n-1)! \left(\frac{\partial}{\partial t} A_g^{n-1} g + A_g^n g \right). \end{aligned}$$

Now

$$\begin{aligned} A_g^{n-1} g &= \left[A_{\Phi^t}^{(1)^{-1}} \boldsymbol{\nabla} \left(A_g^{n-2} g \right) \right] \cdot \boldsymbol{\nabla} g \\ &= \boldsymbol{\nabla} \varphi_n \cdot \boldsymbol{\nabla} g \end{aligned}$$

where

$$\boldsymbol{\nabla}\varphi_n = \left[A_{\Phi^t}^{(1)^{-1}} \boldsymbol{\nabla} \left(A_g^{n-2} g \right) \right].$$

We obtain,

$$\frac{\partial}{\partial t} \boldsymbol{\nabla}\varphi_n = A_{\Phi^t}^{(1)^{-1}} \left(\frac{\partial}{\partial t} \boldsymbol{\nabla} A_g^{n-2} g + \mathbf{Hess} g \boldsymbol{\nabla}\varphi_n - \boldsymbol{\nabla} g \cdot \boldsymbol{\nabla} \left(\boldsymbol{\nabla}\varphi_n \right) \right).$$

We then have

$$\begin{aligned}
\frac{\partial}{\partial t} A_g^{n-1} g &= \frac{\partial}{\partial t} \boldsymbol{\nabla}\varphi_n \cdot \boldsymbol{\nabla} g \\
&= \left[A_{\Phi^t}^{(1)^{-1}} \left(\frac{\partial}{\partial t} \boldsymbol{\nabla} A_g^{n-2} g + \mathbf{Hess} g \boldsymbol{\nabla}\varphi_n - \boldsymbol{\nabla} g \cdot \boldsymbol{\nabla} \left(\boldsymbol{\nabla}\varphi_n \right) \right) \right] \cdot \boldsymbol{\nabla} g \\
&= \left[A_{\Phi^t}^{(1)^{-1}} \left(\frac{\partial}{\partial t} \boldsymbol{\nabla} A_g^{n-2} g + \boldsymbol{\nabla} \left(\boldsymbol{\nabla}\varphi_n \cdot \boldsymbol{\nabla} g \right) \right) \right] \cdot \boldsymbol{\nabla} g \\
&= A_g \left[\frac{\partial}{\partial t} A_g^{n-2} g + A_g \left(A_g^{n-2} g \right) \right] \\
&= A_g H_g \left(A_g^{n-2} g \right). \\
&= A_g H_g \left(\frac{1}{(n-2)!} H_g^{(n-2)} g \right) \qquad \text{(from the induction hypothesis)} \\
&= \frac{1}{(n-2)!} A_g H_g^{(n-1)} g \\
&= \frac{1}{(n-2)!} A_g \left((n-1)! A_g^{n-1} g \right) \qquad \text{(still by the induction hypothesis)} \\
&= (n-1) A_g^n g.
\end{aligned}$$

Thus,

$$\begin{aligned}
H_g^n g &= (n-1)! \, (n-1+1) \, A_g^n g \\
&= n! A_g^n g
\end{aligned}$$

■

Proposition 4 *If $g(0) = 0$, then the formula*

$$\theta_\Lambda^{(n)}(t) = (n-1)! < A_g^{n-1} g >_{t,\Lambda}, \quad n \geq 2$$

still holds if we no longer require Ψ and g to be compactly supported in $\mathbb{R}^\Lambda$.

Proof. As in [8], consider the family cutoff functions

$$\chi = \chi_\varepsilon \tag{34}$$

($\varepsilon \in [0,1]$) in $\mathcal{C}_o^\infty(\mathbb{R})$ with value in $[0,1]$ such that

$$\begin{cases} \chi = 1 & \text{for } |t| \leq \varepsilon^{-1} \\ \left| \chi^{(k)}(t) \right| \leq C_k \dfrac{\varepsilon}{|t|^k} & \text{for } k \in \mathbb{N} \end{cases}$$

We could take for instance

$$\chi_\varepsilon(t) = f(\varepsilon \ln |t|)$$

for a suitable f.

We then introduce

$$\Psi_\varepsilon(x) = \chi_\varepsilon(|x|)\Psi, \qquad x \in \mathbb{R}^\Lambda \tag{35}$$

and

$$g_\varepsilon(x) = \chi_\varepsilon(|x|)g \qquad x \in \mathbb{R}^\Gamma \tag{36}$$

One can check that both $\Psi_\varepsilon(x)$ and $g_\varepsilon(x)$ satisfies the assumptions made above on Ψ and g. Now consider the equation

$$-\boldsymbol{\Delta} f_\varepsilon + \boldsymbol{\nabla}\Phi^t_\varepsilon \cdot \boldsymbol{\nabla} f_\varepsilon = g_\varepsilon - < g_\varepsilon >_{t,\Lambda}. \tag{37}$$

which implies

$$\left(-\boldsymbol{\Delta} + \boldsymbol{\nabla}\Phi^t_\varepsilon \cdot \boldsymbol{\nabla}\right) \otimes \mathbf{v}_\varepsilon + \mathbf{Hess}\Phi^t_\varepsilon \mathbf{v}_\varepsilon = \boldsymbol{\nabla} g_\varepsilon \tag{38}$$

where

$$\mathbf{v}_\varepsilon = \boldsymbol{\nabla} f_\varepsilon$$

It was proved in [8] that $\mathbf{v}_\varepsilon = A^{(1)^{-1}}_{\Phi^t} \boldsymbol{\nabla} g_\varepsilon$ converges in C^∞ to $A^{(1)^{-1}}_{\Phi^t} \boldsymbol{\nabla} g$ as $\varepsilon \to 0$. ■

Remark 5 *If we denote by R_n the remainder of the Taylor series expansion of the pressure $P_\Lambda(t)$, given by*

$$P_\Lambda(t) = \frac{\theta_\Lambda(t)}{|\Lambda|}$$

we have

$$\begin{aligned} R_n &= \frac{P^{(n+1)}_\Lambda(t_o)}{(n+1)!} \\ &= \left.\frac{< A^n_g g >_{t,\Lambda}}{(n+1)\,|\Lambda|}\right|_{t=t_o}. \end{aligned}$$

If Φ and g are such that $< A^n_g g >_{t,\Lambda}$ is uniformly bounded with respect to n and does not grow faster than $|\Lambda|$, we automatically get the analyticity of the pressure in the thermodynamic limit.

5 Some Consequences of the Formula for $nth-$Derivative of the Pressure.

In the following, we shall additionally assume that

$$\begin{gathered} \boldsymbol{\nabla} g(0) = 0, \quad \text{and} \\ \boldsymbol{\nabla}\Phi^t_\Lambda(0) = 0 \quad \text{for all } t \in [0, T). \end{gathered}$$

When $n=1$, we recall that $A_g^0 g = g$,

$$\theta'_\Lambda(t) = < g >_{t,\Lambda}$$

and if

$$\mathbf{v}(t) = \boldsymbol{\nabla} f = A_{\Phi^t}^{(1)^{-1}} \boldsymbol{\nabla} g,$$

then we have

$$\left(-\boldsymbol{\Delta} + \boldsymbol{\nabla}\Phi_\Lambda^t \cdot \boldsymbol{\nabla}\right) \otimes \mathbf{v}(t) + \mathbf{Hess}\Phi_\Lambda^t \mathbf{v}(t) = \boldsymbol{\nabla} g$$

and as in [8] $\mathbf{v}(t)$ is a solution of the equation

$$g = < g >_{t,\Lambda} + \mathbf{v}(t) \cdot \boldsymbol{\nabla}\Phi_\Lambda^t - div\mathbf{v}(t). \tag{39}$$

Using the assumptions above, we have

$$\begin{aligned} \theta'_\Lambda(t) &= < g >_{t,\Lambda} \\ &= div\mathbf{v}(t)(0). \end{aligned}$$

Similarly, the formula

$$\theta_\Lambda^{(n)}(t) = (n-1)! < A_g^{n-1} g >_{t,\Lambda},$$

implies that

$$\theta_\Lambda^{(n)}(t) = (n-1)! div\mathbf{v}_n(t)(0),$$

where

$$\mathbf{v}_n(t) = A_{\Phi^t}^{(1)^{-1}} \boldsymbol{\nabla}\left(A_g^{n-1} g\right).$$

Remark 6 *The idea of representing $\theta'_\Lambda(t)$ in terms of $div\mathbf{v}(t)(0)$ is due to Helffer and Sjöstrand [8] in the context of proving the exponential convergence of the thermodynamic limit in the one dimensional case.*

We conclude these notes by a discussion about the potential contribution of this results towards solving the two dimensional dipole gas problem. The dipole gas and other gases of particles interacting through Coulomb forces are very important statistical systems. In particular, for dipole gas, the lack of screening is well known [59], and the analyticity of the pressure in the high temperature and low activity region has been proved in an indirect way, by means of renormalization group methods (see [60] and [61]).
A direct proof of the analyticity of the pressure based on estimating the coefficients of the Mayer (Taylor) series is still an open problem. The close relationship between this model and the Coulomb gas in the Kostelitz-Thouless phase ($\beta > 8\pi$), go along with the non-existence of any proof for the analyticity of the pressure in the Coulomb gas. Indirect arguments are attempted in [62],[63] and [64]. We believe that after a suitable regularization of the Coulomb potential at short distances to assure stability, we can fit the problem into the framework of

the model discussed above and get an estimate of the coefficients of the Mayer series through our formula for the nth derivative of the pressure.

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